Multiscale reduction of discrete Korteweg–de Vries equations

2009 J. Phys. A: Math. Theor. 42 454018

(http://iopscience.iop.org/1751-8121/42/45/454018)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 454018 (18pp) doi:10.1088/1751-8113/42/45/454018

Multiscale reduction of discrete Korteweg–de Vries equations

C Scimiterna

Dipartimento di Fisica e Ingegneria Elettronica, Università degli Studi Roma tre,
Via della Vasca Navale 84, 00146 Roma, Italy

E-mail: scimiterna@fis.uniroma3.it

Received 19 March 2009
Published 27 October 2009
Online at stacks.iop.org/JPhysA/42/454018

Abstract

We show how through a multiscale reduction technique, performing the analysis at orders beyond the nonlinear Schrödinger equation, one can effectively prove if some nonlinear partial difference equation is not integrable. The example is carried out on a symmetric discretization of the KdV equation and is compared to a similar reduction performed on the integrable lattice potential KdV equation.

PACS numbers: 02.30.Jr, 02.30.Ik

1. Introduction

Since the work of Zakharov and Kuznetsov [21], it has become clear how to guess by a perturbative multiscale analysis if a given nonlinear partial differential equation (PDE) is integrable or not. A guiding principle in this approach is that, if the multiscale analysis is applied in the proper way, *from an integrable system one must get another integrable system*. So if, starting from an integrable PDE, we find a previously unknown system, this last will be certainly integrable. Or, if we perform a multiscale analysis of a model whose integrability property is not *a priori* known and at a certain order during the expansion we find a nonintegrable system, we can conclude that the starting model is not integrable. These considerations opened the way to classify all the first nontrivial reduced systems one could obtain in a multiscale analysis and to study the related implications concerning the integrability of the starting model [1–6]. In this way, these first nontrivial reduced systems were elevated to the status of models of a somehow *universal* nature. One of them of particular ubiquitous character is the well-known nonlinear Schrödinger equation (NLS), which is an evolution integrable by the inverse scattering transform. The first attempt to go beyond the NLS order has been presented by Santini, Degasperis and Manakov in [8]. The authors, only demanding that the series representing the perturbative solution of our problem would remain at any order a *uniformly* valid approximation to the real solution, succeeded in removing from the expansion all the diverging terms, the so-called *secularities*. With this at hand Degasperis and

Procesi in [9, 10, 18] finally developed a secularity-free integrability test for a real dispersive nonlinear PDE. In more recent times in [11–13], the theory developed to test the PDE has been applied to the case of real dispersive partial difference equations ($P\Delta E$). The basic ingredients of this approach are as follows:

- the linear part of the $P\Delta E$ has the plane wave solution $e^{i[\kappa n-\omega(\kappa)m]}$, characterized by the dispersion relation $\omega = \omega(\kappa)$. Function $\omega(\kappa)$ is an analytic function of κ around $\kappa = 0$ so that it admits the Taylor series representation

$$\omega(\kappa) = \sum_{j=0}^{+\infty} \omega_j(0)\kappa^j, \qquad \omega_j(\kappa) \doteq \frac{1}{j!}\frac{d^j}{d\kappa^j}\omega(\kappa); \tag{1}$$

- solution $u_{n,m}$ to a given $P\Delta E$ has a $\mathcal{C}^{(\infty)}$ dependence on a set of *slow-time* variables $m_j \doteq M_j\varepsilon^j m$, $j \geqslant 1$ and on a *slow-space* variable $n_1 \doteq N_1\varepsilon n$, where $\varepsilon > 0$ is the perturbative parameter and M_j, $j \geqslant 1$ and N_1 are real constants possibly such that n_1, m_j, $j \geqslant 1$ turn out to be integers. This $\mathcal{C}^{(\infty)}$ dependence allows us to express any n or m shift in the $P\Delta E$ as differential operators involving the derivatives with respect to the slow variables;
- the real solution $u(n, m, n_1, \{m_j\}_{j\geqslant 1}, \varepsilon)$ was chosen so as to be represented by the perturbative series

$$\begin{aligned} u(n, m, n_1, \{m_j\}_{j\geqslant 1}, \varepsilon) &= \sum_{\gamma=1}^{+\infty}\sum_{\alpha=-\gamma}^{\gamma} \varepsilon^\gamma u_\gamma^{(\alpha)}(n_1, \{m_j\}_{j\geqslant 1})E_{n,m}^\alpha, \\ E_{n,m} \doteq e^{i[\kappa n-\omega(\kappa)m]}, &\qquad u_\gamma^{(-\alpha)} = \bar{u}_\gamma^{(\alpha)}; \end{aligned} \tag{2}$$

for more details see [20]. In [11], the multiscale analysis of the well-known integrable lattice potential KdV equation (lpKdV)

$$\mu(u_{n+1,m+1} - u_{n,m}) + \zeta(u_{n+1,m} - u_{n,m+1}) = (u_{n+1,m} - u_{n,m+1})(u_{n+1,m+1} - u_{n,m}), \tag{3}$$

was performed until order ε^3 at which one finds an integrable NLS equation, while in [12] it was shown that the straightforward discretization of the KdV equation

$$u_{n,m+1} - u_{n,m-1} = \frac{\alpha}{4}(u_{n+3,m} - 3u_{n+1,m} + 3u_{n-1,m} - u_{n-3,m}) - \frac{\beta}{2}\left(u_{n+1,m}^2 - u_{n-1,m}^2\right), \tag{4}$$

gives rise to an integrable NLS equation, too. In this paper, we will carry out the multiscale analysis of equations (3) and (4) up to order ε^5 thus showing that at this order the lpKdV equation passes the integrability test while equation (4) does not.

This paper is organized as follows. In section 2, we will recall the main results on the integrability test based on multiscale analysis contained in the works of Santini, Degasperis and Manakov [8] and in those by Degasperis and Procesi [9, 10, 18]. In particular, we will recall the notion of *asymptotic integrability* and all the asymptotic integrability conditions which have to be satisfied to pass the test at a certain order. Here we will present for the first time the so-called A_3 integrability conditions. Then in sections 3 and 4, after we recall the results contained in [11, 12] about the reduction of equations (3) and (4) until the NLS order, we will examine the higher orders in the reduction process, thus giving an analytic evidence of the nonintegrability of equation (4).

2. The orders beyond the NLS, equations and the integrability conditions

Let us emphasize the fundamental role covered by the higher orders in setting up an integrability test for nonlinear differential equations. The importance of the following considerations is in the fact that everything we will say here remains the same even if we consider nonlinear partial difference or differential-difference systems. The first attempt to go beyond the NLS order has been presented in [8] and the authors, starting from *S*-integrable models, through a combination of an asymptotic functional analysis and spectral methods, succeeded in removing all the secular terms from the reduced equations they found order by order. Their findings could be summarized as follows:

- the number of slow-time variables required for amplitudes $u_n^{(\alpha)}$ coincides with the number of nonvanishing coefficients $\omega_n(\kappa)$ defined in equation (1);
- amplitude $u_1^{(1)}$ evolves at the slow-times $t_n, n \geqslant 3$ according to the nth equation of the NLS hierarchy;
- amplitudes $u_m^{(1)}, m \geqslant 2$ evolve at the slow-times $t_n, n \geqslant 2$ according to certain *linear, nonhomogeneous* equations supplemented by some *asymptotic conditions* on functions $u_p^{(1)}, p \geqslant 2$ themselves.

Thus, one can conclude that the cancellation at each stage of the perturbation process of all the secular terms from the reduced equations is a sufficient request to uniquely fix the evolution equations followed by every $u_n^{(1)}, n \geqslant 1$ at each slow-time. The result in the second point should be expected as a hierarchy of integrable equations always represents a set of compatible evolutions for a unique function u at different times, or the equations in this hierarchy are *generalized symmetries* of each other; for more details see [7, 19].

Although this procedure provides the most general *necessary and sufficient* conditions to get secularity-free reduced equations, it is not necessary to maintain such a functional approach to develop an integrability test. A *recursive* technique proves to be more suitable. As illustrated in [9, 10, 18] the authors, through a detailed multiscale reduction of the spectral problem associated with an *S*-integrable equation or of the linearizing process associated with a *C*-integrable system, showed the following.

Theorem 1. *If a nonlinear PDE is (C or S) integrable, then under a multiscale expansion, functions $u_m^{(1)}, m \geqslant 1$ satisfy the equations*

$$\partial_{t_n} u_1^{(1)} = K_n\big[u_1^{(1)}\big], \tag{5a}$$

$$\mathcal{M}_n u_j^{(1)} = f_n(j), \qquad \mathcal{M}_n \doteq \partial_{t_n} - K_n'\big[u_1^{(1)}\big], \qquad \forall j, \quad n \geqslant 2, \tag{5b}$$

where $K_n\big[u_1^{(1)}\big]$ is the nth flow in the nonlinear Schrödinger hierarchy and $f_n(j)$ is a inhomogeneous nonlinear forcing term . All the other $u_m^{(\kappa)}, \kappa \geqslant 2$ are expressed in terms of differential monomials of $u_\rho^{(1)}, \rho \leqslant m$.

In other words, integrability is a *sufficient* condition for harmonics $u_n^{(1)}, n \geqslant 1$ to satisfy equations (5), or equations (5) are a *necessary* condition for integrability. In equations (5*b*), $K_n'[u]v$ is the Frechet derivative of nonlinear term $K_n[u]$ along direction v defined by

$$K_n'[u]v \doteq \frac{\mathrm{d}}{\mathrm{d}s} K_n[u + sv]|_{s=0},$$

i.e., the linearization near u of $K_n[u]$ along direction v. If $K_n[u]$ depends explicitly on $x, t, u, u_x, u_{xx}\ldots, \bar{u}, \bar{u}_x, \bar{u}_{xx}, \ldots,$ the explicit expression of $K_n'[u]v$ is

$$K_n'[u]v = \frac{\partial K_n}{\partial u} v + \frac{\partial K_n}{\partial u_x} v_x + \frac{\partial K_n}{\partial u_{xx}} v_{xx} + \cdots + \frac{\partial K_n}{\partial \bar{u}} \bar{v} + \frac{\partial K_n}{\partial \bar{u}_x} \bar{v}_x + \frac{\partial K_n}{\partial \bar{u}_{xx}} \bar{v}_{xx} + \cdots.$$

For future use, we note that operator $K_n'[u]$ is a *linear* operator when it acts on a linear combination of functions with *real* coefficients. Equations (5*a*) represent a hierarchy of *compatible* evolutions for the same function $u_1^{(1)}$ at different slow-times. Those evolutions are characterized by the commutativity condition

$$\mathcal{M}_r K_s - \mathcal{M}_s K_r = 0. \tag{6}$$

In contrast, as we will see, the compatibility of equations (5*b*) is not always guaranteed but is subject to a sort of commutativity conditions among their rhs terms $f_n(j)$'s. These last commutativity conditions will be the cornerstone of our integrability test. We have the following.

Theorem 2. *Operators $\mathcal{M}_m$ defined in equation (5b) commute among themselves. Once we fix index $j \geqslant 2$ in the set of equations (5b), theorem 2 implies the following compatibility conditions:*

$$\mathcal{M}_k f_n(j) = \mathcal{M}_n f_k(j), \qquad \forall\, k, n \geqslant 2, \tag{7}$$

where $f_n(j)$ and $f_k(j)$ are functions of the fundamental harmonic $u_m^{(1)}$ with $1 \leqslant m \leqslant j-1$. The time derivatives ∂_{t_k}, ∂_{t_n} of $u_m^{(1)}$ appearing in $\mathcal{M}_k$ and $\mathcal{M}_n$ respectively, have to be eliminated using the evolution equations (5a), (5b) up to index $j-1$.

Let us continue illustrating the results of Degasperis *et al*. As relations (5) imply an infinite number of *asymptotic symmetries* for the PDE under investigation, Degasperis *et al* [10] stated the following.

Conjecture 1. *If a PDE admits a multiscale expansion with functions $u_m^{(1)}$, $m \geqslant 1$ satisfying equations (5) $\forall\, j, n \geqslant 2$, then the equation is (C or S) integrable.*

In other words, the conjecture affirms that the relations (5) are a *sufficient* condition for integrability or that integrability is a *necessary* condition to have a multiscale expansion where equations (5) are satisfied. Let us consider the following definitions:

Definition 1. *A differential monomial $\rho\big[u_j^{(1)}\big]$, $j \geqslant 1$, in the functions $u_j^{(1)}$, their complex conjugates and their ξ-derivatives are of 'gauge' 1 if the following transformation property,*

$$\rho\big[\tilde{u}_j^{(1)}\big] = \mathrm{e}^{\mathrm{i}\theta}\rho\big[u_j^{(1)}\big], \qquad \tilde{u}_j^{(1)} \doteq \mathrm{e}^{\mathrm{i}\theta}u_j^{(1)},$$

is valid;

Definition 2. *The order of a differential monomial $\rho\big[u_j^{(1)}\big]$, $j \geqslant 1$, in the functions $u_j^{(1)}$, their complex conjugates and their ξ-derivatives are*

$$order\,\big(\partial_\xi^m u_j^{(1)}\big) = order\,\big(\partial_\xi^m \bar{u}_j^{(1)}\big) = m + j, \qquad m \geqslant 0;$$

Definition 3. *A finite-dimensional vector space $\mathcal{P}_n$, $n \geqslant 2$ is the set of all differential polynomials of gauge 1 and order n in the functions $u_j^{(1)}$, $j \geqslant 1$, their complex conjugates and their ξ-derivatives;*

Definition 4. *$\mathcal{P}_n(m)$, $m \geqslant 1$ and $n \geqslant 2$ is the subspace of $\mathcal{P}_n$ whose elements are differential polynomials of gauge 1 and order n in the functions $u_j^{(1)}$, their complex conjugates and their ξ-derivatives with $1 \leqslant j \leqslant m$.*

From definition (4) one has that $\mathcal{P}_n = \mathcal{P}_n(n-2)$ and moreover one can see that in general $K_n[u_1^{(1)}] \in \partial_\xi^n u_1^{(1)} \cup \mathcal{P}_{n+1}(1)$ and that $f_n(j) \in \mathcal{P}_{j+n}(j-1)$, where $(j, n) \geqslant 2$. The basis of differential monomials of spaces $\mathcal{P}_n(m)$ can be found in [20]. We have the following theorem [9].

Theorem 3. *If for each fixed $j \geqslant 2$ equation (7) with $k = 2$ and $n = 3$, namely $\mathcal{M}_2 f_3(j) = \mathcal{M}_3 f_2(j)$, is satisfied, then there exist unique differential polynomials $f_n(j)$ $\forall n \geqslant 4$ such that the flows $\mathcal{M}_n u_j^{(1)} = f_n(j)$ commute for any $n \geqslant 2$.*

Hence among the relations (7) only those with $k = 2$ and $n = 2$ have to be tested. The following theorem ensures that the multiscale expansion (5) is *secularity-free.*

Theorem 4. *The homogeneous equation $\mathcal{M}_n u = 0$ has no solution u in vector space $\mathcal{P}_m$, i.e., $Ker(\mathcal{M}_n) \cap \mathcal{P}_m = \emptyset$.*

Finally, we define the *degree of integrability* of a given equation.

Definition 5. *If the relations (7) are satisfied up to index j, $j \geqslant 2$, we say that our equation is asymptotically integrable of degree j or A_j integrable.*

2.1. The integrability conditions for the NLS hierarchy

In this subsection, we will present the conditions for asymptotic integrability of order n or A_n integrability conditions for $n = 1, 2, 3$. To simplify the notation, we will use for $u_j^{(1)}$ the concise form $u(j)$. First, for future convenience, we list the fluxes $K_n[u]$ of the NLS hierarchy up to $n = 4$:

$$K_1[u] \doteq A u_\xi, \tag{8a}$$

$$K_2[u] \doteq -\mathrm{i}\rho_1 \left[u_{\xi\xi} + \frac{\rho_2}{\rho_1} |u|^2 u \right], \tag{8b}$$

$$K_3[u] \doteq B \left[u_{\xi\xi\xi} + \frac{3\rho_2}{\rho_1} |u|^2 u_\xi \right], \tag{8c}$$

$$K_4[u] \doteq -\mathrm{i}C \left\{ u_{\xi\xi\xi\xi} + \frac{\rho_2}{\rho_1} \left[\frac{3\rho_2}{2\rho_1} |u|^4 u + 4|u|^2 u_{\xi\xi} + 3u_\xi^2 \bar{u} + 2|u_\xi|^2 u + u^2 \bar{u}_{\xi\xi} \right] \right\}, \tag{8d}$$

and the corresponding $K_n'[u]v$ up to $n = 3$:

$$K_1'[u]v = A v_\xi, \tag{9a}$$

$$K_2'[u]v = -\mathrm{i}\rho_1 \left\{ v_{\xi\xi} + \frac{\rho_2}{\rho_1} [u^2 \bar{v} + 2|u|^2 v] \right\}, \tag{9b}$$

$$K_3'[u]v = B \left\{ v_{\xi\xi\xi} + \frac{3\rho_2}{\rho_1} [|u|^2 v_\xi + \bar{u} u_\xi v + u u_\xi \bar{v}] \right\}, \tag{9c}$$

where A, ρ_1, ρ_2, B and C are arbitrary real constants.

The A_1 integrability condition is given by the reality of coefficient ρ_2 of the nonlinear term in the NLS equation. It is obtained by commuting the NLS flux $K_2[u]$, where ρ_1 and ρ_2 are supposed complex, with the most general flux that belongs to the same vector space of $K_3[u]$. This vector space is $\partial_\xi^3 u_1^{(1)} \cup \mathcal{P}_4(1)$ and the flux is $B[u_{\xi\xi\xi} + \tau|u|^2 u_\xi + \mu u^2 \bar{u}_\xi]$, with B, τ and μ being complex constants. Let us remark again that, if we start from an integrable model, the resulting NLS equation should be integrable as well and, as an integrable equation, it should be part of a hierarchy of equations like (5*a*). This commutativity condition gives, if $\rho_2 \neq 0$,

$$\mathrm{Im}[\rho_2] = \mathrm{Im}[B] = \mathrm{Im}[\rho_1] = 0, \qquad \tau = 3\rho_2/\rho_1, \qquad \mu = 0. \tag{10}$$

If $\rho_2 = 0$, it follows $\tau = \mu = 0$ and no conditions on B and ρ_1 (although in general they are real).

The A_2 integrability conditions [9, 10, 18] are obtained choosing $j = 2$ in the compatibility conditions (7) with $k = 2$ and $n = 3$

$$\mathcal{M}_2 f_3(j) = \mathcal{M}_3 f_2(j). \tag{11}$$

In this case, we have that $f_2(2) \in \mathcal{P}_4(1)$ and $f_3(2) \in \mathcal{P}_5(1)$ with $\dim(\mathcal{P}_4(1)) = 2$ and $\dim(\mathcal{P}_5(1)) = 5$, so that from the basis of monomials we derive that $f_2(2)$ and $f_3(2)$ have the form

$$f_2(2) \doteq a u_\xi(1)|u(1)|^2 + b\bar{u}_\xi(1)u(1)^2, \tag{12a}$$

$$f_3(2) \doteq \alpha|u(1)|^4 u(1) + \beta|u_\xi(1)|^2 u(1) + \gamma u_\xi(1)^2\bar{u}(1) + \delta\bar{u}_{\xi\xi}(1)u(1)^2 + \epsilon|u(1)|^2 u_{\xi\xi}(1), \tag{12b}$$

characterized by 2 and 5 complex constants. If $\rho_2 \neq 0$, eliminating from equation (11) the derivatives of $u(1)$ with respect to slow-times t_2 and t_3 given by the evolutions (5a) with $n = 2$ and $n = 3$, we obtain two A_2 integrability conditions

$$a = \bar{a}, \qquad b = \bar{b}. \tag{13}$$

If $\rho_2 = 0$, we have no conditions on a and b. The expressions of $\alpha, \beta, \gamma, \delta, \epsilon$ in terms of a and b are

$$\alpha = \frac{3\mathrm{i}B\rho_2 a}{4\rho_1^2}, \qquad \beta = \frac{3\mathrm{i}Bb}{\rho_1}, \gamma = \frac{3\mathrm{i}Ba}{2\rho_1}, \qquad \delta = 0, \qquad \epsilon = \gamma. \tag{14}$$

The A_3 integrability conditions are derived in a similar way setting $j = 3$ in equation (11). In this case, we have that $f_2(3) \in \mathcal{P}_5(2)$ and $f_3(3) \in \mathcal{P}_6(2)$ with $\dim(\mathcal{P}_5(2)) = 12$ and $\dim(\mathcal{P}_6(2)) = 26$, so that $f_2(3)$ and $f_3(3)$ will depend, respectively, on 12 and 26 complex constants

$$\begin{aligned} f_2(3) \doteq{}& \tau_1|u(1)|^4u(1) + \tau_2|u_\xi(1)|^2u(1) + \tau_3|u(1)|^2u_{\xi\xi}(1) + \tau_4\bar{u}_{\xi\xi}(1)u(1)^2 + \tau_5u_\xi(1)^2\bar{u}(1) \\ &+ \tau_6u_\xi(2)|u(1)|^2 + \tau_7\bar{u}_\xi(2)u(1)^2 + \tau_8u(2)^2\bar{u}(1) + \tau_9|u(2)|^2u(1) \\ &+ \tau_{10}u(2)u_\xi(1)\bar{u}(1) + \tau_{11}u(2)\bar{u}_\xi(1)u(1) + \tau_{12}\bar{u}(2)u_\xi(1)u(1), \end{aligned} \tag{15a}$$

$$\begin{aligned} f_3(3) \doteq{}& \gamma_1|u(1)|^4u_\xi(1) + \gamma_2|u(1)|^2u(1)^2\bar{u}_\xi(1) + \gamma_3|u(1)|^2u_{\xi\xi\xi}(1) + \gamma_4u(1)^2\bar{u}_{\xi\xi\xi}(1) \\ &+ \gamma_5|u_\xi(1)|^2u_\xi(1) + \gamma_6\bar{u}_{\xi\xi}(1)u_\xi(1)u(1) + \gamma_7u_{\xi\xi}(1)\bar{u}_\xi(1)u(1) \\ &+ \gamma_8u_{\xi\xi}(1)u_\xi(1)\bar{u}(1) + \gamma_9|u(1)|^4u(2) + \gamma_{10}|u(1)|^2u(1)^2\bar{u}(2) + \gamma_{11}\bar{u}_\xi(1)u(2)^2 \\ &+ \gamma_{12}u_\xi(1)|u(2)|^2 + \gamma_{13}|u_\xi(1)|^2u(2) + \gamma_{14}|u(2)|^2u(2) + \gamma_{15}u_\xi(1)^2\bar{u}(2) \\ &+ \gamma_{16}|u(1)|^2u_{\xi\xi}(2) + \gamma_{17}u(1)^2\bar{u}_{\xi\xi}(2) + \gamma_{18}u(2)\bar{u}_{\xi\xi}(1)u(1) + \gamma_{19}u(2)u_{\xi\xi}(1)\bar{u}(1) \\ &+ \gamma_{20}\bar{u}(2)u_{\xi\xi}(1)u(1) + \gamma_{21}u(2)u_\xi(2)\bar{u}(1) + \gamma_{22}\bar{u}(2)u_\xi(2)u(1) \\ &+ \gamma_{23}u_\xi(2)u_\xi(1)\bar{u}(1) + \gamma_{24}u_\xi(2)\bar{u}_\xi(1)u(1) + \gamma_{25}\bar{u}_\xi(2)u_\xi(1)u(1) \\ &+ \gamma_{26}\bar{u}_\xi(2)u(2)u(1). \end{aligned} \tag{15b}$$

First, we eliminate from equation (11) with $j = 3$ the derivatives of $u(1)$ with respect to slow-times t_2 and t_3 using the evolutions (5a) respectively with $n = 2$ and $n = 3$ and the same derivatives of $u(2)$ using the evolutions (5b) with $n = 2$ and $n = 3$. Then, indicating with R_i and I_i the real and imaginary parts of τ_i, $i = 1, \ldots, 12$, we obtain the A_3 integrability conditions when $\rho_2 \neq 0$

$$
\begin{aligned}
&R_1 = -\frac{aI_6}{4\rho_1}, \qquad R_3 = \frac{(b-a)I_6}{2\rho_2} - \frac{aI_{12}}{2\rho_2}, \qquad R_4 = \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{aI_{12}}{4\rho_2},\\
&R_5 = \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{(2b-a)I_{12}}{4\rho_2}, \qquad R_6 = -\frac{aI_8}{\rho_2}, \qquad R_7 = R_{12} + \frac{(a-b)I_8}{\rho_2},\\
&R_8 = R_9 = 0, \qquad R_{10} = R_{12}, \qquad R_{11} = R_{12} + \frac{(a-2b)I_8}{\rho_2},\\
&I_4 = \frac{(b+a)R_{12}}{4\rho_2} + \frac{\rho_1 I_1}{\rho_2} + \frac{I_2 - I_3 - 2I_5}{4} + \frac{[2b(a-b)+a^2]I_8}{4\rho_2^2}, \qquad I_7 = 0,\\
&I_9 = 2I_8, \qquad I_{10} = I_{12}, \qquad I_{11} = I_6 + I_{12}.
\end{aligned}
\tag{16}
$$

Although in [9, 10], it was already reported that these conditions would consist of 15 real equations so that $f_2(3)$ and $f_3(3)$ will be parametrized by $2 \times 12 - 15 = 9$ real constants, the precise form of those equations was not given and it appeared in [20] for the first time. For completeness, we give the expressions of γ_j, $j = 1, \ldots, 26$ as functions of τ_i, $i = 1, \ldots, 12$:

$$
\begin{aligned}
&\gamma_1 = \frac{3B}{8\rho_1^2}\left[-2bR_{12} - 8\rho_1 I_1 + 2(I_2 - 2I_3 - 2I_5)\rho_2 + \mathrm{i}(b-5a)I_6 + \frac{2a^2 I_8}{\rho_2} - 3\mathrm{i}aI_{12}\right],\\
&\gamma_2 = -\frac{3Ba}{4\rho_1^2}\left[\mathrm{i}I_6 + \frac{(a-2b)I_8}{\rho_2} + \tau_{12}\right], \qquad \gamma_3 = \frac{3\mathrm{i}B\tau_3}{2\rho_1}, \qquad \gamma_4 = 0, \qquad \gamma_5 = \frac{3\mathrm{i}B\tau_2}{2\rho_1},\\
&\gamma_6 = \frac{3\mathrm{i}B\tau_4}{\rho_1}, \qquad \gamma_7 = \gamma_5, \qquad \gamma_8 = \gamma_3 + \frac{3\mathrm{i}B\tau_5}{\rho_1}, \qquad \gamma_9 = -\frac{3B(\rho_2 I_6 + 3a\mathrm{i}I_8)}{4\rho_1^2},\\
&\gamma_{10} = \frac{3\mathrm{i}B\rho_2 R_6}{2\rho_1^2}, \qquad \gamma_{11} = 0, \qquad \gamma_{12} = \frac{3\mathrm{i}B\tau_9}{2\rho_1},\\
&\gamma_{13} = \frac{3\mathrm{i}B\tau_{11}}{2\rho_1}, \qquad \gamma_{14} = 0, \qquad \gamma_{15} = \frac{3\mathrm{i}B\tau_{12}}{2\rho_1},\\
&\gamma_{16} = \frac{3\mathrm{i}B\tau_6}{2\rho_1}, \qquad \gamma_{17} = \gamma_{18} = 0, \qquad \gamma_{19} = \frac{3\mathrm{i}B\tau_{10}}{2\rho_1}, \qquad \gamma_{20} = \gamma_{15}, \qquad \gamma_{21} = \frac{3\mathrm{i}B\tau_8}{\rho_1},\\
&\gamma_{22} = \gamma_{12}, \qquad \gamma_{23} = \gamma_{16} + \gamma_{19}, \qquad \gamma_{24} = \gamma_{13}, \qquad \gamma_{25} = \frac{3\mathrm{i}B\tau_7}{\rho_1}, \qquad \gamma_{26} = 0.
\end{aligned}
\tag{17}
$$

If $\rho_2 = 0$, the A_3 integrability conditions turn out to be

$$
\begin{aligned}
&\tau_1 = -\frac{\mathrm{i}}{4\rho_1}[b(\tau_{11} - 2\tau_6) + \bar{a}\tau_7], \qquad \bar{b}\tau_7 = \frac{1}{2}(b-a)(\tau_{11} + \tau_{10} - \tau_6) + \bar{a}\tau_7,\\
&a\tau_8 = b\tau_8 = 0, \qquad a\tau_9 = b\tau_9 = 0, \qquad \bar{a}\tau_{12} = a(\tau_{10} - \tau_{11}) + b\tau_6 + \bar{a}\tau_7,\\
&(\bar{b} - \bar{a})\tau_{12} = (b-a)\tau_{10},
\end{aligned}
\tag{18}
$$

and the expressions of γ_j as functions of τ_i are

$$
\gamma_1 = -\frac{3B}{4\rho_1^2}(a\tau_6 - 4\mathrm{i}\rho_1\tau_1 + \bar{b}\tau_{12}), \qquad \gamma_2 = -\frac{3B}{4\rho_1^2}(b\tau_6 + \bar{a}\tau_7). \tag{19}
$$

Other γ_j are given in equation (17) (note that, given the conditions (18), from the expressions of γ_9 and γ_{10} one deduces that $\gamma_9 = \gamma_{10} = 0$). Also in this case the conditions given in equations (18) appear to be new. Their importance resides in the fact that a C-integrable equation must satisfy those conditions (in this case, equation (8*b*) is linear and corresponds to set $\rho_2 = 0$).

3. Multiscale reduction of the lattice potential KdV equation (lpKdV)

Here we present the multiscale reduction of the lpKdV equation (3) up to order ε^5, thus showing that this equation is (at least) A_3 integrable. In [11], the lpKdV equation was shown to be (at least) A_1 integrable. Let us briefly review the main results of this reduction.

(i) Order $\gamma = 1$.

- $\alpha = 0$: at this order, equation (3) is automatically satisfied;
- $\alpha = 1$: one obtains the *dispersion relation*

$$e^{-i\omega} = \frac{\mu - \zeta e^{i\kappa}}{\mu e^{i\kappa} - \zeta}, \tag{20}$$

which, solved, gives

$$\omega(\kappa) = 2 \arctan\left(\frac{\zeta + \mu}{\mu - \zeta} \tan\frac{\kappa}{2}\right); \tag{21}$$

(ii) Order $\gamma = 2$.

- $\alpha = 1$:

$$(\epsilon\partial_{n_1} - \partial_{m_1})u_1^{(1)} = 0, \qquad \epsilon \doteq -\frac{N_1}{M_1}\omega_1. \tag{22}$$

Defining

$$N_1 \doteq \pm S e^{-i\omega}(\mu e^{i\kappa} - \zeta), \qquad M_1 \doteq -S e^{i\kappa}(\mu e^{-i\omega} + \zeta), \qquad S \in \mathbb{C}, \tag{23}$$

one obtains $\epsilon = \pm 1$, which means that

$$u_1^{(1)}\big(n_1, \{m_j\}_{j=1}^K\big) = u_1^{(1)}\big(n_2, \{m_j\}_{j=2}^K\big), \qquad n_2 \doteq n_1 + \epsilon m_1 \tag{24}$$

(with ω_1 defined in equation (1)). The complex constant $S \doteq r e^{i\theta}$, $r > 0$, is to be chosen so that $\theta = -\arctan[\zeta \sin\kappa/(\zeta\cos\kappa - \mu)]$ in such a way that N_1 and M_1 are indeed real numbers. Taking into account the dispersion relation (20), N_1 and M_1 can be rewritten as

$$N_1 = \epsilon S(\mu - \zeta e^{i\kappa}), \qquad M_1 = S e^{i\kappa}\frac{\zeta^2 - \mu^2}{\mu e^{i\kappa} - \zeta}; \tag{25}$$

- $\alpha = 0$:

$$\partial_{n_2} u_1^{(0)} = \alpha_1 \big|u_1^{(1)}\big|^2; \tag{26}$$

- $\alpha = 2$:

$$u_2^{(2)} = \alpha_2 u_1^{(1)2}, \qquad \alpha_2 \doteq \frac{1 + e^{i\kappa}}{(1 - e^{i\kappa})(\mu + \zeta)}; \tag{27}$$

(iii) Order $\gamma = 3$.

- $\alpha = 1$:

$$(\partial_{n_1} - \epsilon\partial_{m_1})u_2^{(1)} = 0, \tag{28a}$$

$$i\partial_{m_2} u_1^{(1)} = \rho_1 \partial_{n_2}^2 u_1^{(1)} + \rho_2 u_1^{(1)}\big|u_1^{(1)}\big|^2, \qquad -\frac{2\rho_2}{\rho_1} = \alpha_1^2, \tag{28b}$$

$$\rho_1 \doteq \frac{\mu\zeta M_1^2 \sin\kappa}{M_2(\mu^2 - \zeta^2)} = -\frac{N_1^2}{M_2}\omega_2, \qquad \rho_2 \doteq \frac{8\zeta\mu(\zeta - \mu)(1 + \cos\kappa)^2 \sin\kappa}{M_2(\mu + \zeta)(\zeta^2 + \mu^2 - 2\mu\zeta\cos\kappa)^2}. \tag{28c}$$

The first relation says that $u_2^{(1)}$ depends on n_2 too while the second one is an NLS equation giving the evolution of $u_1^{(1)}$ according to slow-time m_2.

Theorem 5. *Equation (28b), whose coefficients are defined in (28c), is an integrable (continuous, defocusing) nonlinear Schrödinger equation, its integrability arising from the manifest reality of its coefficients. This proves the A_1 asymptotic integrability of the $lpKdV$ equation.*

From the above NLS equation (28*b*), one derives the *continuity equation*

$$\partial_{m_2} d^{(1)} = \rho_1 \partial_{n_2} J_2^{(1)}, \qquad d^{(1)} \doteq \left|u_1^{(1)}\right|^2, \qquad J_2^{(1)} \doteq -\mathrm{i}\left(\bar{u}_1^{(1)} \partial_{n_2} u_1^{(1)} - \mathcal{C.C.}\right), \tag{29}$$

where we used symbols $d^{(1)}$ and $J_2^{(1)}$ to indicate that those quantities represent, respectively, a density of a conserved quantity and a current density. Differentiating by m_2 equation (26), using continuity equation (29) and integrating with respect to n_2 setting equal to zero the arbitrary n_2-independent integration function (all the $u_n^{(\alpha)}$ s go to zero as $n_2 arrow \pm\infty$), we get

$$\partial_{m_2} u_1^{(0)} = \alpha_1 \rho_1 J_2^1. \tag{30}$$

Equation (30) will be used in the following subsections.

3.1. Higher orders in the multiscale expansion of the $lpKdV$ equation

We now give a detailed exposition of the multiscale analysis at the orders beyond the NLS scale. To do so, we need to present first the behavior of the higher order harmonics at this order.

- $\alpha = 0$: taking into consideration equations (20), (25), (26), (30) and the fact that $u_1^{(0)}$ and $u_1^{(1)}$ depend on n_2 and choosing $u_2^{(0)}$ as a field depending on n_2, we obtain

$$\partial_{n_2} u_2^{(0)} = d^{(2)}, \qquad d^{(2)} \doteq \alpha_1\left(u_1^{(1)} \bar{u}_2^{(1)} + \mathcal{C.C.}\right) + \rho_3 J_2^{(1)}, \qquad \rho_3 \doteq \frac{2\sin(\kappa)}{(\mu+\zeta)}, \tag{31}$$

  where we introduced symbol $d^{(2)}$ to indicate that this expression represents another density of a conserved quantity;
- $\alpha = 2$: taking into consideration equations (20), (25), (27) and the fact that both $u_1^{(1)}$ and $u_2^{(2)}$ depend on n_2, we have

$$u_3^{(2)} = u_1^{(1)}\left[\alpha_3 \partial_{n_2} u_1^{(1)} + 2\alpha_2 u_2^{(1)}\right], \qquad \alpha_3 \doteq \frac{2\epsilon S\,\mathrm{e}^{\mathrm{i}\kappa}(\mu - \zeta\,\mathrm{e}^{\mathrm{i}\kappa})}{(\mathrm{e}^{\mathrm{i}\kappa}-1)^2(\mu+\zeta)} = \frac{2\mathrm{i}N_1\alpha_2}{(\mu+\zeta)\rho_3}; \tag{32}$$

- $\alpha = 3$: using equations (20) and (27), we obtain

$$u_3^{(3)} = \alpha_2^2 u_1^{(1)3}. \tag{33}$$

With these results at hand we can go over to the higher order.

(iv) Order $\gamma = 4$.

- $\alpha = 1$: taking into account equations (20), (25)–(27), (28*b*), (30)–(32), that $u_1^{(0)}, u_2^{(0)}, u_1^{(1)}, u_2^{(1)}$ and $u_2^{(2)}$ depend on n_2 and that (see sections 2 and 2.1) amplitude $u_1^{(1)}$ evolves at slow-time m_3 according to the *complex modified KdV equation* ($cmKdV$)

$$\partial_{m_3} u_1^{(1)} - B\left(\partial_{n_2}^3 u_1^{(1)} + \frac{3\rho_2}{\rho_1}\left|u_1^{(1)}\right|^2 \partial_{n_2} u_1^{(1)}\right) = 0, \tag{34}$$

  we get

$$(\partial_{n_1} - \epsilon\partial_{m_1}) u_3^{(1)} = \mathcal{N}_2\left(u_1^{(1)}, u_2^{(1)}\right). \tag{35}$$

Here, $\mathcal{N}_2\big(u_1^{(1)}, u_2^{(1)}\big)$ is a nonlinear function in $u_1^{(1)}$ and its complex conjugate, and a linear function in $u_2^{(1)}$ and its complex conjugate. As seen before in equations (24), (28*a*), the rhs of equation (35) depends on n_2 so that it is in the kernel of the linear operator on the lhs and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the rhs and the lhs be equal to zero. We obtain

$$(\partial_{n_1} - \epsilon\partial_{m_1})u_3^{(1)} = 0, \tag{36a}$$

$$\partial_{m_2}u_2^{(1)} - K_2'\big[u_1^{(1)}\big]u_2^{(1)} = \mathcal{N}_2^1\big(u_1^{(1)}\big), \tag{36b}$$

$$K_2'\big[u_1^{(1)}\big]u_2^{(1)} \doteq -\mathrm{i}\rho_1\left[\partial_{n_2}^2 u_2^{(1)} + \frac{\rho_2}{\rho_1}\big(u_1^{(1)2}\bar{u}_2^{(1)} + 2\big|u_1^{(1)}\big|^2 u_2^{(1)}\big)\right].$$

The first relation shows that $u_3^{(1)}$ itself depends on n_2. In the second relation, which comes directly from $\mathcal{N}_2\big(u_1^{(1)}, u_2^{(1)}\big) = 0$, $\mathcal{N}_2^1\big(u_1^{(1)}\big)$ is another nonlinear function involving only $u_1^{(1)}$ and its complex conjugate and $K_2'\big[u_1^{(1)}\big]u_2^{(1)}$ is the Frechet derivative of the NLS flux $K_2\big[u_1^{(1)}\big]$ (see section 2.1). Term $\mathcal{N}_2^1\big(u_1^{(1)}\big)$ depends on the free real constant B appearing in equation (34). Choosing coefficient B so as to eliminate any dependence in the resulting equation on $\partial_{n_2}^3 u_1^{(1)}$, we obtain

$$\partial_{m_2}u_2^{(1)} - K_2'\big[u_1^{(1)}\big]u_2^{(1)} = bu_1^{(1)2}\partial_{n_2}\bar{u}_1^{(1)} + a\big|u_1^{(1)}\big|^2\partial_{n_2}u_1^{(1)}, \tag{37a}$$

$$a \doteq -N_1\rho_2\cot\kappa, \qquad b \doteq a\frac{2-\cos\kappa}{\cos\kappa}, \qquad b = a - 2\rho_1\rho_3\alpha_1 \tag{37b}$$

$$\begin{aligned} B &= \frac{\epsilon\mu\zeta M_1^3}{3M_3(\zeta^2-\mu^2)^2}[(\mu^2+\zeta^2+2\mu\zeta\cos\kappa)\cos\kappa - 4\mu\zeta] \\ &= \frac{N_1^3}{M_3}\omega_3. \end{aligned} \tag{37c}$$

A particular solution of the nonhomogeneous equation (37*a*) is given by

$$\mathcal{S}_{\text{part.}} = -\mathrm{i}\frac{a}{2\rho_1}u_1^{(1)}\int_{\xi_0}^{n_2} w\big|u_1^{(1)}(n_2')\big|^2 dn_2' + \mathrm{i}\frac{b-a}{2\rho_2}\partial_{n_2}u_1^{(1)}, \tag{38}$$

where ξ_0 is a real constant. The elimination of any term of form $\partial_{n_2}^3 u_1^{(1)}$ from the rhs of equation (37*a*) is justified from the following theorem:

Theorem 6. *If a function $q(x, t_r, t_s)$ evolves according to equation $\partial_{t_r}q - K_r[q] = 0$ and if $K_s[q]$ is such that $[K_r, K_s]_L = 0$ (cfr. equation (6)), then term $\partial_{t_s}q - K_s[q]$ is secular for equation $\big(\partial_{t_r} - K_r'[q]\big)\phi(x, t_r) = f_r(x, t_r, q, q_x, \ldots)$, where $f_r(x, t_r, q, q_x, \ldots)$ is a generic forcing term and $\phi(x, t_r)$ is a generic function of its arguments.*

Proof. It is sufficient to show that $\partial_{t_s}q - K_s[q]$ solves the homogeneous equation. In fact we have $\big(\partial_{t_r} - K_r'[q]\big)\big(\partial_{t_s}q - K_s[q]\big) = \partial_{t_r}\big(\partial_{t_s}q - K_s[q]\big) - K_r'[q]\big(\partial_{t_s}q - K_s[q]\big) = \partial_{t_s}\partial_{t_r}q - \hat{\partial}_{t_r}K_s[q] - K_s'[q]\partial_{t_r}q - K_r'[q]\partial_{t_s}q + K_r'[q]K_s = \partial_{t_s}\partial_{t_r}q - \hat{\partial}_{t_r}K_s[q] - K_s'[q]K_r - \partial_{t_s}K_r[q] + \hat{\partial}_{t_s}K_r[q] + K_r'[q]K_s = \partial_{t_s}\big(\partial_{t_r}q - K_r[q]\big) - [K_r, K_s]_L = 0.$ □

Notation $\hat{\partial}_{t_j}$ indicates differentiation with respect to a possible *explicit* dependence of various $K_m[q]$ on the slow-times. From the proof it is clear that, when various $K_m[q]$ do not exhibit an explicit dependence on the slow-times, the two terms $\partial_{t_s}q$ and $K_s(q)$ are indeed *separately secular*. If the rhs of equation (37*a*) contains a term of form $\partial_{n_2}^3 u_1^{(1)}$, it is always possible to evidence in it a term of form $K_3\big[u_1^{(1)}\big]$, the flux of the $cmKdV$ equation (34), which, from the above theorem is secular.

Theorem 7. *The coefficients of the rhs of equation (37a) given by equation (37b), obviously satisfy the A_2 integrability conditions (13). This proves the A_2 asymptotic integrability of the $lpKdV$ equation.*

Let us now present the results we get at order $\gamma = 4$ for the other harmonics. From equations (29), (34) and (28b), (31), (37a) we get respectively the continuity equations

$$\partial_{m_3} d^{(1)} = B\partial_{n_2} J_3^{(1)}, \tag{39a}$$

$$\partial_{m_2} d^{(2)} = \partial_{n_2} J_2^{(2)}. \tag{39b}$$

As before with $J_3^{(1)}$ and $J_2^{(2)}$ we have indicated the current densities related respectively to densities $d^{(1)}$ and $d^{(2)}$

$$J_3^{(1)} \doteq \left(\frac{3\rho_2}{2\rho_1}\big|u_1^{(1)}\big|^4 + u_1^{(1)}\partial_{n_2}^2\bar{u}_1^{(1)} - \big|\partial_{n_2}u_1^{(1)}\big|^2 + \bar{u}_1^{(1)}\partial_{n_2}^2 u_1^{(1)}\right), \tag{39c}$$

$$\begin{aligned} J_2^{(2)} \doteq{}& -\rho_1\rho_3\big(\bar{u}_1^{(1)}\partial_{n_2}^2 u_1^{(1)} - 2\big|\partial_{n_2}u_1^{(1)}\big|^2 + u_1^{(1)}\partial_{n_2}^2\bar{u}_1^{(1)}\big) + \left[(a+b)\frac{\alpha_1}{2} - \rho_2\rho_3\right]\big|u_1^{(1)}\big|^4 \\ & + i\rho_1\alpha_1\big(u_1^{(1)}\partial_{n_2}\bar{u}_2^{(1)} + u_2^{(1)}\partial_{n_2}\bar{u}_1^{(1)} - \mathcal{C.C.}\big). \end{aligned} \tag{39d}$$

Combining equations (39a) and (39b) with equations (26) and (31) respectively, give the relations (as usual the arbitrary n_2-independent integration functions have been set to zero to match the asymptotic conditions on $u_n^{(\alpha)}$)

$$\partial_{m_3} u_1^{(0)} = \alpha_1 B J_3^{(1)}, \qquad \partial_{m_2} u_2^{(0)} = J_2^{(2)}, \tag{40}$$

necessary for the following computations:

- $\alpha = 0$: taking into account equations (20), (25), (26), (27), (28b), (30), (31), (40), that $u_1^{(0)}, u_2^{(0)}, u_1^{(1)}, u_2^{(1)}$ depend on n_2 and choosing $u_3^{(0)}$ dependent on n_2, we have

$$\partial_{n_2} u_3^{(0)} = d^{(3)}, \tag{41}$$

$$\begin{aligned} d^{(3)} \doteq{}& -\mathrm{i}\rho_3\big(\bar{u}_2^{(1)}\partial_{n_2}u_1^{(1)} + \bar{u}_1^{(1)}\partial_{n_2}u_2^{(1)} - \mathcal{C.C.}\big) + \alpha_1\big(u_1^{(1)}\bar{u}_3^{(1)} + \bar{u}_1^{(1)}u_3^{(1)} + |u_2^{(1)}|^2\big) \\ & + \frac{3M_1}{2(\alpha-\beta)}\alpha_1^2\big|u_1^{(1)}\big|^4 + \frac{M_1^2}{12}\alpha_1\Bigg(\bar{u}_1^{(1)}\partial_{n_2}^2u_1^{(1)} + u_1^{(1)}\partial_{n_2}^2\bar{u}_1^{(1)} \\ & + \frac{4\sin^2(\kappa/2) - 1}{\cos^2(\kappa/2)}\big|\partial_{n_2}u_1^{(1)}\big|^2\Bigg); \end{aligned}$$

- $\alpha = 2$: taking into account equations (20), (25), (26), (27), (28b), (32), (33) and that $u_1^{(0)}, u_1^{(1)}, u_2^{(1)}, u_2^{(2)}$ and $u_3^{(2)}$ depend on n_2, we have

$$\begin{aligned} u_4^{(2)} ={}& 2\alpha_2^3(1 + 4\sin^2(\kappa/2))\big|u_1^{(1)}\big|^2 u_1^{(1)2} + \frac{\alpha_3^2}{2\alpha_2}\big[\big(\partial_{n_2}u_1^{(1)}\big)^2 + u_1^{(1)}\partial_{n_2}^2u_1^{(1)}\cos\kappa\big] \\ & + \alpha_2\big(2u_1^{(1)}u_3^{(1)} + u_2^{(1)2}\big) + \alpha_3\partial_{n_2}\big(u_2^{(1)}u_1^{(1)}\big); \end{aligned} \tag{42}$$

- $\alpha = 3$: taking into account equations (20), (25), (27), (32), (33) and that $u_1^{(1)}, u_2^{(2)}$ and $u_3^{(3)}$ depend on n_2, we have

$$u_4^{(3)} = \alpha_2\big[3\alpha_2u_2^{(1)} + 2\alpha_3\big(\partial_{n_2}u_1^{(1)}\big)\big]u_1^{(1)2}; \tag{43}$$

- $\alpha = 4$: taking into account equations (20), (27), (33) we obtain

$$u_4^{(4)} = \alpha_2^3 u_1^{(1)4}. \tag{44}$$

With these results we can go over to a higher order.

(v) Order $\gamma = 5$.

- $\alpha = 1$: taking into account equations (20), (25)–(27), (28*b*), (30)–(34), (37*a*)–(37*c*), (39*c*), (39*d*), (40)–(42), the fact that $u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_2^{(2)}, u_3^{(2)}$ depend on n_2 and that (see sections 2 and 2.1)

$$\partial_{m_3} u_2^{(1)} - K_3'\big[u_1^{(1)}\big]u_2^{(1)} = f_3(2), \tag{45}$$

$$\partial_{m_4} u_1^{(1)} + \mathrm{i}C\left\{\partial_{n_2}^4 u_1^{(1)} + \frac{\rho_2}{\rho_1}\left[\frac{3\rho_2}{2\rho_1}\big|u_1^{(1)}\big|^4 u_1^{(1)} + 4\big|u_1^{(1)}\big|^2 \partial_{n_2}^2 u_1^{(1)} \right.\right.$$
$$\left.\left. + 3\bar{u}_1^{(1)}\big(\partial_{n_2} u_1^{(1)}\big)^2 + 2\big|\partial_{n_2} u_1^{(1)}\big|^2 u_1^{(1)} + u_1^{(1)2}\partial_{n_2}^2 \bar{u}_1^{(1)}\right]\right\} = 0, \tag{46}$$

where $K_3'[u]v$ is given by equation (9*c*) and $f_3(2)$ by equation (12*b*) and (14), we obtain

$$(\partial_{n_1} - \epsilon\partial_{m_1})u_4^{(1)} = \mathcal{N}_3\big(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\big). \tag{47}$$

Here, $\mathcal{N}_3\big(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\big)$ is a linear function in $u_3^{(1)}$ and its complex conjugate, and a nonlinear function in $u_1^{(1)}$ and $u_2^{(1)}$ and their complex conjugates. As seen before in equations (24), (28*a*), (36*a*), the rhs of equation (47) depends on n_2 so that it is in the kernel of the linear operator on the lhs and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the rhs and the lhs be equal to zero. We obtain

$$\big(\partial_{n_1} - \epsilon\partial_{m_1}\big)u_4^{(1)} = 0, \tag{48a}$$

$$\partial_{m_2} u_3^{(1)} - K_2'\big[u_1^{(1)}\big]u_3^{(1)} = \mathcal{N}_3^1\big(u_1^{(1)}, u_2^{(1)}\big). \tag{48b}$$

The first relation shows that $u_4^{(1)}$ itself depends on n_2. In the second relation, which comes directly from $\mathcal{N}_3\big(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\big) = 0$, $\mathcal{N}_3^1\big(u_1^{(1)}, u_2^{(1)}\big)$ is another nonlinear function involving $u_1^{(1)}, u_2^{(1)}$ and their complex conjugates. Now the term $\mathcal{N}_3^1\big(u_1^{(1)}, u_2^{(1)}\big)$ contains the free real constant C which is chosen so as to eliminate any dependence of the resulting equation on $\partial_{n_2}^4 u_1^{(1)}$. From theorem 6, the presence of this term can always introduce a dependence on the secular term $K_4\big[u_1^{(1)}\big]$, the flux of equation (45). So we obtain

$$\partial_{m_2} u_3^{(1)} - K_2'\big[u_1^{(1)}\big]u_3^{(1)} = f_2(3), \tag{49a}$$

$$C = \frac{\mu\zeta M_1^4[\mu^4 - 20\mu^2\zeta^2 + \zeta^4 + 8\mu\zeta(\mu^2+\zeta^2)\cos\kappa + 2\mu^2\zeta^2\cos(2\kappa)]}{12M_4(\mu^2-\zeta^2)^3}\sin\kappa$$
$$= \frac{N_1^4}{M_4}\omega_4, \tag{49b}$$

where the forcing term $f_2(3)$ is given by equation (15*a*).

Theorem 8. *Term $f_2(3)$ appearing in equation (49a) obviously has all its coefficients that satisfy all the fifteen A_3 integrability conditions (16). This proves the A_3 asymptotic integrability of the lpKdV equation.*

Due to the fact that the 12 complex coefficients of $f_2(3)$ respect the A_3 integrability conditions, they can all be expressed in terms of a convenient nine-dimensional real basis. The base is defined by R_1, I_1, R_2, I_2, I_3, I_5, R_6 and R_{12}, I_{12} (see equation (15*a*)) and given by

$$R_1 = 0, \qquad I_1 = -\rho_2 \frac{(-23 + 16\cos\kappa + \cos(2\kappa))\cot^2(\kappa/2)}{2(\mu+\zeta)^2}, \qquad R_2 = 0$$
$$I_2 = -N_1^2\rho_2 \frac{(29 - 24\cos\kappa + 7\cos(2\kappa))}{12\sin^2\kappa}, \qquad I_3 = -\frac{1}{6}N_1^2(1 + 3\csc^2\kappa)\rho_2, \tag{50}$$
$$I_5 = -\frac{1}{4}N_1^2(1 + 2\csc^2\kappa)\rho_2, \qquad R_6 = R_{12} = a, \qquad I_{12} = 0.$$

4. Multiscale analysis of the symmetric discretization of the KdV equation

Let us briefly review the main results of the multiscale analysis of the symmetric discretization of the KdV equation obtained in [12] until the NLS order.

(i) Order $\gamma = 1$.
- $\alpha = 0$: at this order, equation (4) is automatically satisfied;
- $\alpha = 1$: we get the dispersion relation

$$\sin\omega = \alpha\sin^3\kappa, \tag{51}$$

which is used to express α in terms of κ and ω;
(ii) Order $\gamma = 2$.
- $\alpha = 0$:

$$\partial_{m_1} u_1^{(0)} = 0, \tag{52}$$

and we choose $u_1^{(0)} = 0$;
- $\alpha = 1$:

$$(\epsilon\partial_{n_1} - \partial_{m_1})u_1^{(1)} = 0, \qquad \epsilon \doteq -\frac{N_1}{M_1}\omega_1. \tag{53}$$

Choosing

$$N_1 = \pm S\sin\kappa\cos\omega, \qquad M_1 = -3S\cos\kappa\sin\omega, \tag{54}$$

where S is an arbitrary real constant, one obtains $\epsilon = \pm 1$ so that $u_1^{(1)}$ is a function of $n_2 \doteq n_1 + \epsilon m_1$;
- $\alpha = 2$:

$$u_2^{(2)} = \alpha_2 u_1^{(1)2}, \qquad \alpha_2 \doteq -\frac{\beta\sin(2\kappa)\csc\omega}{4(4\cos^3\kappa - \cos\omega)}; \tag{55}$$

(iii) Order $\gamma = 3$.
- $\alpha = 0$:

$$u_2^{(0)} = \alpha_1 |u_1^{(1)}|^2, \qquad \alpha_1 \doteq \frac{1}{3}\beta\cot\omega\tan\kappa = -\frac{N_1}{M_1}\beta\epsilon; \tag{56}$$

- $\alpha = 1$:

$$(\partial_{n_1} - \epsilon\partial_{m_1})u_2^{(1)} = 0, \tag{57a}$$
$$\mathrm{i}\partial_{m_2} u_1^{(1)} = \rho_1\partial_{n_2}^2 u_1^{(1)} + \rho_2 u_1^{(1)}|u_1^{(1)}|^2, \tag{57b}$$

$$\rho_1 \doteq -\frac{3S^2}{4M_2}[2+3\cos(2\kappa)-\cos(2\omega)]\tan\omega = -\frac{N_1^2}{M_2}\omega_2,$$
$$\rho_2 \doteq -\frac{\beta^2[5+3\cos(2\kappa)-16\cos^3\kappa\cos\omega+2\cos(2\omega)]\sin\kappa\tan\kappa\csc(2\omega)}{6M_2(4\cos^3 k-\cos\omega)}$$
$$= \frac{(\alpha_1+\alpha_2)\beta\sin\kappa\sec\omega}{M_2}. \tag{57c}$$

Equation (57a) shows that $u_2^{(1)}$ depends on n_2 while equation (57b) is an NLS equation giving the evolution of $u_1^{(1)}$ in slow-time m_2.

Theorem 9. *As ρ_2 is a real number, the A_1 integrability condition, equation (10), is satisfied and the obtained NLS equation is integrable. Hence our starting model (4) is A_1-integrable.*

4.1. Higher orders in the multiscale expansion of the symmetric discretization of the KdV equation

We now present the multiscale analysis at orders beyond the NLS scale. As everything resembles the case of the lpKdV equation, we will state only the results thereby showing the nonintegrability of this equation.

(iii) Order $\gamma = 3$.

- $\alpha = 2$:

$$u_3^{(2)} = \alpha_3 u_1^{(1)}\partial_{n_2}u_1^{(1)} + 2\alpha_2 u_1^{(1)}u_2^{(1)},$$
$$\alpha_3 \doteq -\frac{\mathrm{i}S\epsilon\beta\sin\kappa\{[1+16\cos^3\kappa\cos\omega-2\cos(2\omega)]\cos(2\kappa)-3\cos(2\omega)\}}{4(4\cos^3\kappa-\cos\omega)^2\sin\omega}; \tag{58}$$

- $\alpha = 3$:

$$u_3^{(3)} = \alpha_2\alpha_4 u_1^{(1)3}, \qquad \alpha_4 \doteq -\frac{\beta\sin(3\kappa)}{\{[1+2\cos(2\kappa)]^3+4\sin^2\omega-3\}\sin\omega}; \tag{59}$$

(iv) Order $\gamma = 4$.

- $\alpha = 0$:

$$u_3^{(0)} = \alpha_1\left(u_1^{(1)}\bar{u}_2^{(1)} + \mathcal{C.C.}\right) + \rho_3 J_2^{(1)},$$
$$J_2^{(1)} \doteq -\mathrm{i}\left(\bar{u}_1^{(1)}\partial_{n_2}u_1^{(1)} - \mathcal{C.C.}\right),$$
$$\rho_3 \doteq -\frac{S\epsilon\beta[2+3\cos(2\kappa)-\cos(2\omega)]\csc\omega\sec\kappa\tan\kappa}{12} = -\frac{\epsilon M_2\alpha_1\rho_1}{M_1}; \tag{60}$$

- $\alpha = 1$:

$$\partial_{m_2}u_2^{(1)} - K_2'\left[u_1^{(1)}\right]u_2^{(1)} = b u_1^{(1)2}\partial_{n_2}\bar{u}_1^{(1)} + a\left|u_1^{(1)}\right|^2\partial_{n_2}u_1^{(1)}, \tag{61a}$$

$$a \doteq -\frac{3BM_3\rho_2}{M_2\rho_1} + 6N_1\rho_2\cot\kappa\tan^2\omega + \frac{\beta\sec\omega[2N_1(\alpha_1+\alpha_2)\cos\kappa + (\mathrm{i}\alpha_3+\rho_3)\sin\kappa]}{M_2}, \tag{61b}$$

$$b \doteq \frac{\beta \sec\omega[-N_1(\alpha_1+\alpha_2)\cos\kappa + \rho_3\sin\kappa] - 3N_1M_2\rho_2\cot\kappa\tan^2\omega}{M_2},$$
$$B \doteq \frac{\epsilon S^3\{21 + 18\cos(2\kappa)[2-\cos(2\omega)] - 32\cos(2\omega) + \cos(4\omega)\}}{8M_3}$$
$$\cdot\cos\kappa\sec\omega\tan\omega = \frac{N_1^3}{M_3}\omega_3. \tag{61c}$$

Theorem 10. *The coefficients of the rhs of equation (61a), given by equation (61b), respect the two A_2 integrability conditions (13). This proves the A_2 asymptotic integrability of the symmetrically discretized KdV equation.*

- $\alpha = 2$:

$$\begin{aligned} u_4^{(2)} &= 2\alpha_2^2(\alpha_1+\alpha_4+\alpha_5\rho_2)\big|u_1^{(1)}\big|^2u_1^{(1)2} + \big[\alpha_3\delta + \alpha_2\big(\pi + N_1^2\big)\big]\big(\partial_{n_2}u_1^{(1)}\big)^2 \\ &\quad + \big[\alpha_3\delta + \alpha_2\big(\pi + N_1^2 + 2\rho_1\alpha_2\alpha_5\big)\big]u_1^{(1)}\partial_{n_2}^2u_1^{(1)} + \alpha_2\big(2u_1^{(1)}u_3^{(1)} + u_2^{(1)2}\big) \\ &\quad + \alpha_3\partial_{n_2}\big(u_2^{(1)}u_1^{(1)}\big), \qquad \alpha_5 \doteq -\frac{2M_2\cos(2\omega)\csc(2\kappa)}{\beta}, \end{aligned} \tag{62}$$
$$\delta \doteq \frac{3\mathrm{i}\epsilon S\cos\kappa[4\cos\kappa\cos\omega\cos(2\kappa) - \cos(2\omega)]}{2(4\cos^3\kappa - \cos\omega)},$$
$$\pi \doteq \frac{3S^2\cos\kappa\cos\omega[\cos\omega + 3\cos\omega\cos(4\kappa) + 6\cos\kappa\sin^2\omega]}{2(4\cos^3\kappa - \cos\omega)};$$

- $\alpha = 3$:

$$\begin{aligned} u_4^{(3)} &= \alpha_4\big[3\alpha_2u_2^{(1)} + (\alpha_2F+\alpha_3)\big(\partial_{n_2}u_1^{(1)}\big)\big]u_1^{(1)2}, \\ F &\doteq \frac{24\mathrm{i}\epsilon S\cos^3\kappa\cos\omega[4\cos^3(2\kappa) - \cos(2\omega)]}{[1+2\cos(2\kappa)][3+6\cos(2\kappa)+3\cos(4\kappa)+\cos(6\kappa)-\cos(2\omega)]}; \end{aligned} \tag{63}$$

- $\alpha = 4$:

$$\begin{aligned} u_4^{(4)} &= G\alpha_2(\alpha_2+2\alpha_4)u_1^{(1)4}, \\ G &\doteq \frac{\beta\csc\omega\sin(4\kappa)}{4\{-4[\cos\kappa+\cos(3\kappa)]^3 + \cos\omega + \cos(3\omega)\}}; \end{aligned} \tag{64}$$

(v) Order $\gamma = 5$.

- $\alpha = 0$:

$$\begin{aligned} u_4^{(0)} &= -\mathrm{i}\rho_3\big(\bar u_2^{(1)}\partial_{n_2}u_1^{(1)} + \bar u_1^{(1)}\partial_{n_2}u_2^{(1)} - \mathcal{C.C.}\big) + \alpha_1\big(\bar u_1^{(1)}u_3^{(1)} + u_1^{(1)}\bar u_3^{(1)} + \big|u_2^{(1)}\big|^2\big) \\ &\quad + f\big|u_1^{(1)}\big|^4 + g\big|\delta_{n_2}u_1^{(1)}\big|^2 + h\big(u_1^{(1)}\partial_{n_2}^2\bar u_1^{(1)} + \mathcal{C.C.}\big), \\ f &\doteq -\frac{\epsilon}{M_1}\left\{\left[(a+b)\frac{\alpha_1}{2} - \rho_2\rho_3\right]M_2 + \frac{N_1\beta\big(\alpha_1^2+2\alpha_2^2\big)}{2} + \frac{3BM_3\alpha_1\rho_2}{2\rho_1}\right\}, \\ g &\doteq -\frac{-3BM_3\alpha_1 + N_1^3(\beta - 6\alpha\alpha_1) + M_1^3\alpha_1\epsilon + 6M_2\rho_1\rho_3}{3M_1\epsilon}, \\ h &\doteq -\frac{6BM_3\alpha_1 + N_1^3(\beta - 6\alpha\alpha_1) + M_1^3\alpha_1\epsilon - 6M_2\rho_1\rho_3}{6M_1\epsilon}; \end{aligned} \tag{65}$$

- $\alpha = 1$:

$$\partial_{m_2} u_3^{(1)} - K_2'\left[u_1^{(1)}\right] u_3^{(1)} = f_2(3), \tag{66a}$$

$$\begin{aligned} C \doteq{} & \frac{S^4\{-3[404 + 549\cos(2\kappa) + 126\cos(4\kappa)]\cos(2\omega).}{128 M_4} \\ & + \frac{.3[73 + 78\cos(2\kappa) + 9\cos(4\kappa)]\cos(4\omega) - [2 + \cos(2\kappa)]\cos(6\omega).}{128 M_4} \\ & + \frac{.997 + 1358\cos(2\kappa) + 405\cos(4\kappa)\}\sec^2\omega\tan\omega}{128 M_4} = \frac{N_1^4}{M_4}\omega_4, \end{aligned} \tag{66b}$$

where the forcing term $f_2(3)$ is given by equation (15a). The real and imaginary parts of the coefficients τ_i, $i = 1\ldots, 12$ of $f_2(3)$ are given by

$$\begin{aligned} R_1 ={}& 0, \\ I_1 ={}& -\frac{2\beta\rho_1\sin\kappa\left[12N_1\alpha_2^2\alpha_5\rho_1\sec\omega + (3BM_3\alpha_1 - 2M_2\rho_1\rho_3)\csc\omega\tan\kappa\right]}{12N_1M_2\rho_1^2}\rho_2 \\ & - \frac{3aBM_3}{4M_2\rho_1^2}\rho_2 + \left(\frac{3CM_4}{2M_2\rho_1^2} - \frac{1}{2}M_2\tan\omega\right)\rho_2^2 - \frac{\beta\alpha_2^2(2\alpha_1 + 3\alpha_4)\sin\kappa\sec\omega}{M_2} \\ & - \frac{\beta\left[(a+b)M_2\alpha_1 + N_1\left(\alpha_1^2 + 2\alpha_2^2\right)\beta\right]\csc\omega\tan\kappa\sin\kappa}{6N_1M_2}, \\ R_2 ={}& 0, \\ I_2 ={}& \frac{2\left[CM_4 + 9N_1^2M_2\rho_1(1 - \csc^2\kappa + \cot^2\kappa\sec^2\omega) - M_2^2\rho_1^2\tan\omega\right]}{M_2\rho_1}\rho_2 \\ & - \frac{9BM_3N_1\cot\kappa\tan^2\omega}{M_2\rho_1}\rho_2 - 3\left[\frac{bBM_3}{M_2\rho_1} + (a+2b)N_1\cot\kappa\tan^2\omega\right] \\ & - \frac{N_1^2\csc\omega\sin\kappa\tan\kappa}{9M_2}\beta^2 - \left(\frac{N_1\cos\kappa\sec\omega}{M_2} + \frac{2\rho_1\csc\omega\sin\kappa\tan\kappa}{3N_1}\right)\beta\rho_3 \\ & + \frac{N_1\sec\omega(\mathrm{i}\alpha_3\cos\kappa - 2N_1\alpha_2\sin\kappa)}{M_2}\beta + \frac{BM_3\csc\omega\sin\kappa\tan\kappa}{3N_1M_2}\beta\alpha_1 \\ & + \frac{2\csc\kappa\sec\kappa + 3\sec\omega(3\cos\kappa\cot\kappa\tan^2\omega - 2\sin\kappa)}{3M_2}N_1^2\beta\alpha_1, \\ R_3 ={}& 0, \\ I_3 ={}& -\frac{N_1^2\csc\omega\sin\kappa\tan\kappa}{18M_2}\beta^2 - \frac{BM_3\csc\omega\sin\kappa\tan\kappa}{3N_1M_2}\beta\alpha_1 \\ & + \frac{2\csc\kappa\sec\kappa + 3\sec\omega(3\cos\kappa\cot\kappa\tan^2\omega - 2\sin\kappa)}{6M_2}N_1^2\beta\alpha_1 \\ & + \frac{\beta\sec\omega\left\{N_1(\mathrm{i}\alpha_3 + \rho_3)\cos\kappa - \left[\alpha_3\delta + \alpha_2\left(2N_1^2 + \pi + 2\alpha_2\alpha_5\rho_1\right)\right]\sin\kappa\right\}}{M_2} \\ & + \frac{\beta\rho_1\rho_3\csc\omega\sin\kappa\tan\kappa}{3N_1} - 3a\left[\frac{BM_3}{2M_2\rho_1} + N_1\cot\kappa\tan^2\omega\right] \\ & + \frac{4CM_4 - 2M_2^2\rho_1^2\tan\omega + 9N_1(N_1M_2\rho_1\cot\kappa - BM_3)\cot\kappa\tan^2\omega}{M_2\rho_1}\rho_2, \\ R_4 ={}& 0, \end{aligned}$$

$$
\begin{aligned}
I_4 &= -\frac{N_1^2 \csc\omega \sin\kappa \tan\kappa}{18 M_2}\beta^2 + \frac{3}{2} N_1 (2b + 3 N_1 \rho_2 \cot\kappa) \cot\kappa \tan^2\omega \\
&\quad + \frac{C M_4 \rho_2}{M_2 \rho_1} - \frac{B M_3 \csc\omega \sin\kappa \tan\kappa}{3 N_1 M_2}\beta\alpha_1 + \frac{\beta\rho_1\rho_3 \csc\omega \sin\kappa \tan\kappa}{3N_1} \\
&\quad + \frac{2\csc\kappa \sec\kappa + 3\sec\omega (3\cos\kappa \cot\kappa \tan^2\omega - \sin\kappa)}{6M_2} N_1^2 \beta\alpha_1 \\
&\quad - \frac{N_1 (2\rho_3 \cos\kappa + N_1 \alpha_2 \sin\kappa) \sec\omega}{2M_2}\beta, \\
R_5 &= 0, \\
I_5 &= \frac{(\mathrm{i} N_1 \cos\kappa - \delta \sin\kappa) \sec\omega}{M_2}\alpha_3 - \frac{\left[N_1^2(\alpha_1 + 2\alpha_2) + \alpha_2 \pi\right] \sin\kappa \sec\omega}{M_2}\beta \\
&\quad + \frac{N_1 \rho_3 \cos\kappa \sec\omega}{M_2}\beta - M_2 \rho_2 \rho_1 \tan\omega + 3N_1 (3N_1\rho_2 \cot\kappa - a) \cot\kappa \tan^2\omega \\
&\quad + \frac{3(2C M_4 \rho_2 - 6 B M_3 N_1 \rho_2 \cot\kappa \tan^2\omega - a B M_3)}{2 M_2 \rho_1}, \\
\tau_6 &= \tau_7 = a, \qquad \tau_8 = -\mathrm{i}\rho_2, \qquad \tau_9 = -2\mathrm{i}\rho_2, \\
\tau_{10} &= a, \qquad \tau_{11} = 2b, \qquad \tau_{12} = a.
\end{aligned} \tag{67}
$$

Theorem 11. *The coefficients given in equation (67) respect only fourteen out of the fifteen A_3 integrability conditions (16) (the one involving I_4 is not satisfied). This proves that, as expected, the symmetrically discretized KdV equation is not integrable.*

5. Conclusions

We have shown in all details that the symmetric version of the discretized KdV equation is A_2 integrable but not A_3 integrable. This is a very important result as it shows that also in the case of difference equations the A_3 integrability conditions are very restrictive. So also in the case of discrete equations at this order, the integrability conditions are able to distinguish an integrable from a nonintegrable equation. These results confirm the conjected theorem presented in [12] that by multiscale expansion we can effectively prove integrability. Work is in progress to present some results on a possible classification theorem for nonlinear equations on the square.

References

[1] Calogero F 1991 Why are certain nonlinear PDEs both widely applicable and integrable? *What is Integrability?* ed V E Zakharov (Berlin: Springer) pp 1–62

[2] Calogero F and Eckhaus W 1987 Necessary conditions for integrability of nonlinear PDEs *Inverse Problems* **3/2** L27–32

[3] Calogero F and Eckhaus W 1987 Nonlinear evolution equations, rescalings, model PDEs and their integrability: I *Inverse Problems* **3/2** 229–62

[4] Calogero F and Eckhaus W 1988 Nonlinear evolution equations, rescalings, model PDEs and their integrability: II *Inverse Problems* **4/1** 11–33

[5] Calogero F, Degasperis A and X-D Ji 2000 Nonlinear Schrödinger-type equations from multiscale reduction of PDEs: I. Systematic derivation *J. Math. Phys.* **41/9** 6399–443

[6] Calogero F, Degasperis A and X-D Ji 2001 Nonlinear Schrödinger-type equations from multiscale reduction of PDEs: II. Necessary conditions of integrability for real PDEs *J. Math. Phys.* **42/6** 2635–52

[7] Catoni V 1995 Sviluppo a molti tempi in fisica e gerarchia dell'equazione di Schrödinger non lineare *Degree Thesis* physics department, 'La Sapienza' University, Rome

[8] Degasperis A, Manakov S V and Santini P M 1997 Multiple-scale perturbation beyond the nonlinear Schrödinger equation: I *Physica* D **100** 187–211

[9] Degasperis A 2001 Multiscale expansion and integrability of dispersive wave equations. Lectures given at the Euro Summer School *What is Integrability?* (Cambridge: Isaac Newton Institute)
Degasperis A 2009 *Integrability (Lecture Notes in Physics* vol 767) ed A Mikhailov (Berlin: Springer)

[10] Degasperis A and Procesi M 1999 Asymptotic Integrability *in Symmetry and Perturbation Theory, SPT98* ed A Degasperis and G Gaeta (Singapore: World Scientific) pp 23–37

[11] Hernandez Heredero R, Levi D, Petrera M and Scimiterna C 2007 Multiscale expansion of the lattice potential KdV equation on functions of an infinite slow-varyness order *J. Phys. A: Math. Theor.* **40** F831–40

[12] Hernandez Heredero R, Levi D, Petrera M and Scimiterna C 2008 Multiscale expansion on the lattice and integrability of partial difference equations *J. Phys. A: Math. Theor.* **41** 315208–19

[13] Hernandez Heredero R, Levi D, Petrera M and Scimiterna C 2008 Multiscale expansion and integrability properties of the lattice potential KdV equation *J. Nonlinear Math. Phys.* **15/3** 313–23

[14] Kodama Y and Mikhailov A V 1996 Obstacles to asymptotic integrability *Algebraic Aspects of Integrable Systems: in Memory of Irene Dorfman* ed A S Fokas and I M Gelfand (Boston: Birkhäuser) pp 173–204

[15] Levi D and Petrera M 2007 Continuous symmetries of the lattice potential KdV equation *J. Phys. A: Math. Theor.* **40** 4141–59

[16] Mikhailov A V, Shabat A B and Sokolov V V The symmetry approach to classification of integrable equations *What is Integrability?* ed V E Zakharov (Berlin: Springer) pp 115–184

[17] Nijhoff F W and Capel H W 1995 The discrete Korteweg–de Vries equation *Acta Appl. Math.* **39** 133–58

[18] Procesi M *Degree Thesis* physics department, 'La Sapienza' University, Rome

[19] Santini P M 1996 Multiscale expansions, symmetries and the nonlinear Schrödinger hierarchy *Algebraic Aspects of Integrable Systems: in Memory of Irene Dorfman* ed A S Fokas and I M Gelfand (Boston: Birkhäuser) pp 301–13

[20] Scimiterna C *PhD Thesis* physics department 'Roma Tre' University, Rome

[21] Zakharov V E and Kuznetsov E A 1986 Multi-scale expansions in the theory of systems integrable by the inverse scattering transform *Physica* D **18** 455–63